Noodle Bar

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Summer 2020

Think Globally
Act Locally
Eat Noodles

—Lulu, Probably

1 Warm-Ups

Problem 1 (2013 C1). Let n be an positive integer. Find the smallest integer k with the following property; Given any real numbers $a_1, \cdots, a_d$ such that $a_1 + a_2 + \cdots + a_d = n$ and $0 \leq a_i \leq 1$ for $i = 1, 2, \cdots, d$, it is possible to partition these numbers into k groups (some of which may be empty) such that the sum of the numbers in each group is at most 1.

Problem 2 (Hall). Let G be a bipartite graph on A, B. Show that there exists an A-perfect matching (i.e. an injection $f \colon A \to B$ so that $(a, f(a))$ is an edge for all a) if and only if for all subsets S of A, the set

$$\{b \in B \colon (a, b) \text{ is an edge for some } a \in A\}$$

has size at least $|S|$.

2 Problems

Problem 3 (CMC 2020/7). Each of the n^2 cells of an $n \times n$ grid is colored either black or white. Let a_i denote the number of white cells in the i-th row, and let b_i denote the number of black cells in the i-th column. Determine the maximum value of

$$\sum_{i=1}^{n} a_i b_i$$

over all coloring schemes of the grid

Problem 4 (2013 C3). A crazy physicist discovered a new kind of particle wich he called an imon, after some of them mysteriously appeared in his lab. Some pairs of imons in the lab can be entangled, and each imon can participate in many entanglement relations. The physicist has found a way to perform the following two kinds of operations with these particles, one operation at a time.

(i) If some imon is entangled with an odd number of other imons in the lab, then the physicist can destroy it.

(ii) At any moment, he may double the whole family of imons in the lab by creating a copy I' of each imon I. During this procedure, the two copies I' and J' become entangled if and only if the original imons I and J are entangled, and each copy I' becomes entangled with its original imon I; no other entanglements occur or disappear at this moment.

Prove that the physicist may apply a sequence of much operations resulting in a family of imons, no two of which are entangled.

Problem 5 (2014 N3). For each positive integer n, the Bank of Cape Town issues coins of denomination $\frac{1}{n}$. Given a finite collection of such coins (of not necessarily different denominations) with total value at most most $99 + \frac{1}{2}$, prove that it is possible to split this collection into 100 or fewer groups, such that each group has total value at most 1.

Problem 6 (2009 C8). For any integer $n \geq 2$, we compute the integer $h(n)$ by applying the following procedure to its decimal representation. Let r be the rightmost digit of n.

- If $r = 0$, then the decimal representation of $h(n)$ results from the decimal representation of n by removing this rightmost digit 0.
- If $1 \leq r \leq 9$ we split the decimal representation of n into a maximal right part R that solely consists of digits not less than r and into a left part L that either is empty or ends with a digit strictly smaller than r. Then the decimal representation of $h(n)$ consists of the decimal representation of L, followed by two copies of the decimal representation of $R - 1$. For instance, for the number $17, 151, 345, 543$, we will have $L = 17, 151$, $R = 345, 543$ and $h(n) = 17, 151, 345, 542, 345, 542$.

Prove that, starting with an arbitrary integer $n \geq 2$, iterated application of h produces the integer 1 after finitely many steps.

Problem 7 (2006 C4). A cake has the form of an n x n square composed of n^2 unit squares. Strawberries lie on some of the unit squares so that each row or column contains exactly one strawberry; call this arrangement $\mathcal{A}$.

Let $\mathcal{B}$ be another such arrangement. Suppose that every grid rectangle with one vertex at the top left corner of the cake contains no fewer strawberries of arrangement $\mathcal{B}$ than of arrangement $\mathcal{A}$. Prove that arrangement $\mathcal{B}$ can be obtained from $\mathcal{A}$ by performing a number of switches, defined as follows:

A switch consists in selecting a grid rectangle with only two strawberries, situated at its top right corner and bottom left corner, and moving these two strawberries to the other two corners of that rectangle.

Problem 8 (2018 C5). Let k be a positive integer. The organising commitee of a tennis tournament is to schedule the matches for $2k$ players so that every two players play once, each day exactly one match is played, and each player arrives to the tournament site the day of his first match, and departs the day of his last match. For every day a player is present on the tournament, the committee has to pay 1 coin to the hotel. The organisers want to design the schedule so as to minimise the total cost of all players' stays. Determine this minimum cost.

Problem 9 (Turán). Let G be a graph with n vertices and no K_{r+1}. Then, the number of edges in G is maximized when G is a r-partite graph with parts all strictly within 1 of $\frac{n}{r}$.

Problem 10 (2017-2018 USA TST 3). At a university dinner, there are 2017 mathematicians who each order two distinct entrées, with no two mathematicians ordering the same pair of entrées. The cost of each entrée is equal to the number of mathematicians who ordered it, and the university pays for each mathematician's less expensive entrée (ties broken arbitrarily). Over all possible sets of orders, what is the maximum total amount the university could have paid?

Problem 11 (2002 C7). Among a group of 120 people, some pairs are friends. A weak quartet is a set of four people containing exactly one pair of friends. What is the maximum possible number of weak quartets ?