Hallucination

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Abstract

A technique to a specific class of problems. Thanks to Luke Robitaille and Daniel Zhu for useful discussions.

> Choice is an illusion, created between those with power and those without ... Our only hope, our only peace is to understand it, to understand the why.
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> —Merovingian, The Matrix Reloaded

1 Warm-Ups

Exercise 1 (AoPS). Points A_1, B_1, C_1 inside an acute-angled triangle ABC are selected on the altitudes from A, B, C respectively so that the sum of the areas of triangles ABC_1, BCA_1, and CAB_1 is equal to the area of triangle ABC. Prove that the circumcircle of triangle $A_1B_1C_1$ passes through the orthocenter H of triangle ABC.

Exercise 2 (2017 C1). A rectangle $\mathcal{R}$ with odd integer side lengths is divided into small rectangles with integer side lengths. Prove that there is at least one among the small rectangles whose distances from the four sides of $\mathcal{R}$ are either all odd or all even.

Example 3 (2018 C3). Let n be a given positive integer. Sisyphus performs a sequence of turns on a board consisting of $n + 1$ squares in a row, numbered 0 to n from left to right. Initially, n stones are put into square 0, and the other squares are empty. At every turn, Sisyphus chooses any nonempty square, say with k stones, takes one of these stones and moves it to the right by at most k squares (the stone should say within the board). Sisyphus' aim is to move all n stones to square n.

Prove that Sisyphus cannot reach the aim in less than

$$\left\lceil \frac{n}{1} \right\rceil + \left\lceil \frac{n}{2} \right\rceil + \left\lceil \frac{n}{3} \right\rceil + \cdots + \left\lceil \frac{n}{n} \right\rceil$$

turns. (As usual, $\lceil x \rceil$ stands for the least integer not smaller than x.)

2 Historical/Philosophical Interlude

Theorem 4 (Abel-Ruffini-Galois). There exists a degree 5 polynomial whose solutions cannot all be expressed by radicals.

3 Constructive Hallucinations

Problem 5 (Codeforces 1545B). One has some stones on a $1 \times n$ grid. It is allowed to jump one stone over another (i.e. if there is a stone in cell $i-1$ and i, one can move the one on i to $i-2$ or the one on $i-1$ to $i+1$), but one can never have two stones in the same cell. Compute, in linear time in terms of n, the number of possible achievable configurations.

Problem 6 (Codeforces 1270G). Let $a_1, \ldots, a_n$ be integers satisfying

$$i - n \leq a_i \leq i - 1.$$

Show that there exists a nonempty subset S of $\{1, 2, \ldots, n\}$ satisfying

$$\sum_{s \in S} a_i = 0.$$

Problem 7 (2020 USOMO 2+). An empty $2020 \times 2020 \times 2020$ cube is given, and a 2020×2020 grid of square unit cells is drawn on each of its six faces. A beam is a $1 \times 1 \times 2020$ rectangular prism. Several beams are placed inside the cube subject to the following conditions:

- The two 1×1 faces of each beam coincide with unit cells lying on opposite faces of the cube. (Hence, there are $3 \cdot 2020^2$ possible positions for a beam.)
- No two beams have intersecting interiors.
- The interiors of each of the four 1×2020 faces of each beam touch either a face of the cube or the interior of the face of another beam.

Let N be the smallest positive number of beams that can be placed to satisfy these conditions. Compute the number of ways to place exactly N beams to satisfy these conditions.

Problem 8 (2021 C3). Two squirrels, Bushy and Jumpy, have collected 2021 walnuts for the winter. Jumpy numbers the walnuts from 1 through 2021, and digs 2021 little holes in a circular pattern in the ground around their favourite tree. The next morning Jumpy notices that Bushy had placed one walnut into each hole, but had paid no attention to the numbering. Unhappy, Jumpy decides to reorder the walnuts by performing a sequence of 2021 moves. In the k-th move, Jumpy swaps the positions of the two walnuts adjacent to walnut k.

Prove that there exists a value of k such that, on the k-th move, Jumpy swaps some walnuts a and b such that $a < k < b$.

Problem 9 (2018 TSTST 2). In the nation of Onewaynia, certain pairs of cities are connected by one-way roads. Every road connects exactly two cities (roads are allowed to cross each other, e.g., via bridges), and each pair of cities has at most one road between them. Moreover, every city has exactly two roads leaving it and exactly two roads entering it.

We wish to close half the roads of Onewaynia in such a way that every city has exactly one road leaving it and exactly one road entering it. Show that the number of ways to do so is a power of 2 greater than 1 (i.e. of the form 2^n for some integer $n \geq 1$).

4 Miscellaneous Problems

Problem 10 (2017 USAMO 4). Let P_1, P_2, $\ldots$, P_{2n} be $2n$ distinct points on the unit circle $x^2 + y^2 = 1$, other than $(1, 0)$. Each point is colored either red or blue, with exactly n red points and n blue points. Let R_1, R_2, $\ldots$, R_n be any ordering of the red points. Let B_1 be the nearest blue point to R_1 traveling counterclockwise around the circle starting from R_1. Then let B_2 be the nearest of the remaining blue points to R_2 travelling counterclockwise around the circle from R_2, and so on, until we have labeled all of the blue points $B_1, \dots, B_n$. Show that the number of counterclockwise arcs of the form $R_i \to B_i$ that contain the point $(1, 0)$ is independent of the way we chose the ordering $R_1, \dots, R_n$ of the red points.

Problem 11 (EGMO 2016/3). Let m be a positive integer. Consider a $4m \times 4m$ array of square unit cells. Two different cells are related to each other if they are in either the same row or in the same column. No cell is related to itself.Some cells are coloured blue, such that every cell is related to at lest two blue cells. Determine the minimum number of blue cells.

Problem 12 (2016 C7). There are $n \geq 2$ line segments in the plane such that every two segments cross and no three segments meet at a point. Geoff has to choose an endpoint of each segment and place a frog on it facing the other endpoint. Then he will clap his hands $n - 1$ times. Every time he claps,each frog will immediately jump forward to the next intersection point on its segment. Frogs never change the direction of their jumps. Geoff wishes to place the frogs in such a way that no two of them will ever occupy the same intersection point at the same time.

(a) Prove that Geoff can always fulfill his wish if n is odd.
(b) Prove that Geoff can never fulfill his wish if n is even.

Problem 13 (USA TST 2015/3). A physicist encounters 2015 atoms called usamons. Each usamon either has one electron or zero electrons, and the physicist can't tell the difference. The physicist's only tool is a diode. The physicist may connect the diode from any usamon A to any other usamon B. (This connection is directed.) When she does so, if usamon A has an electron and usamon B does not, then the electron jumps from A to B. In any other case, nothing happens. In addition, the physicist cannot tell whether an electron jumps during any given step. The physicist's goal is to isolate two usamons that she is sure are currently in the same state. Is there any series of diode usage that makes this possible?

Problem 14 (APMO 2020/5). Let $n \geq 3$ be a fixed integer. The number 1 is written n times on a blackboard. Below the blackboard, there are two buckets that are initially empty. A move consists of erasing two of the numbers a and b, replacing them with the numbers 1 and $a + b$, then adding one stone to the first bucket and $\gcd(a, b)$ stones to the second bucket. After some finite number of moves, there are s stones in the first bucket and t stones in the second bucket, where s and t are positive integers. Find all possible values of the ratio $\frac{t}{s}$.